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Is Gravitational Dual Charge Physical?

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Abstract

It has been shown that nonzero gravitational dual charge cannot be found in a spin-manifold space-time. Nevertheless, gravitational dual charge is not physically inconsistent with spin-$\frac{1}{2}$, because spinor nodal regions should be excluded from the manifold. However, gravitational dual charge cannot be borne by any object sufficiently localizable that it can be surrounded by an everywhere spacelike two-sphere, hence is not readily visible in Cauchy data.

In 1963 (Lubkin, 1963), I introduced a notion of dual charge that generalizes Dirac's magnetic monopole (Dirac, 1931) to other gauge situations. The context is a bundle B of vector spaces over an underlying base manifold M, with a "gauge" group G of linear transformations acting on the fibers, and bearing a G-admissible parallel displacement – namely, a parallel displacement whose action commutes with the action of G. (It is possible also to generalize to a fiber not explicitly linear.) A mapping D is defined which assigns a "dual-charge" value to each oriented two-sphere in M (i.e., to each continuous map of an oriented two-sphere into M). The dual charge value is a homotopic equivalence class of loops in G, hence it is always trivial either if G is simply connected or if $\pi_2(M)$ is trivial.

Specifically, the map D is defined as follows: The two-sphere is first replaced by a loop l of oriented loops, namely, the circular sections made by the planes through one directed tangent line (refer to a three-dimensional picture of a two-sphere) with the circles oriented in the sense conforming to that of their one common tangent line. The small circles near the beginning of loop l are those whose orientations conform to the infinitesimal-element orientation of the oriented two-sphere. Parallel displacement around a loop in M maps the fiber at the start-finish point onto itself, the linear mapping in question belonging to G. In this way the circular loops are replaced by elements of G, and the loop l of loops by a loop l_G in G. The dual charge that D assigns to the oriented two-sphere is the homotopy class of l_G.

If M is four-dimensional, if B is the tangent bundle, and if G is the Lorentz group L, then we have a context equivalent to the usual $+---$ (or $-+++$) signatured metric of general relativity, "space-time." Since L is not simply connected, the possibility of dual charge may be entertained. It is this possibility that I term "gravitational dual charge." The notion of an L-admissible connection is equivalent to the usual notion of a connection that preserves dot product. The possibility of torsion is irrelevant, because the torsion tensor can be smoothly retracted to zero without affecting the dual charge – i.e., for any torsion-bearing example of gravitational dual charge, a torsionless example can be generated. (This is a comment of D. Finkelstein.) Of course, a torsionless L-admissible connection over a space-time is the unique and familiar Christoffel connection.

Clearly, nontrivial gravitational dual charge over a two-sphere, in requiring the two-sphere not to be homotopic in M to a point, requires the space-time either to have "wormholes," or to have certain points or small regions excluded ("singularities"); M may not be homeomorphic to Minkowski space-time. Is it conceivable that a two-sphere S_2 that in itself is a quite ordinary bag in space-time might surround a physical entity associated with a singular or excluded zone, to produce a gravitational dual charge?

In 1965, D. Finkelstein communicated to me the following proof that this is not possible, provided that the notion "ordinary bag in space-time" refers to a two-sphere $S \subset M$ which can be four-dimensionally thickened to a "shell" neighborhood $S' \subset M$ of S, with S' homeomorphic to $S_2 \otimes$ disk. Indeed, the usual t, x, y, z coordinates for such a shell are carried to a *single* coordinate patch for the shell in M. In such a coordinatization, the metric is represented as an explicit symmetric-matrix function $g_{ij}(t, x, y, z)$. View this as a mapping of the shell S' into the ten-dimensional space of all metrics, M'. Retract the shell in M' to a two-sphere, then retract the two-sphere to a point, this latter being possible because $\pi_2(M')$ can be shown to be trivial. The g_{ij} map should meanwhile be deformed continuously to a constant map, which of course has zero dual charge, hence had zero dual charge originally.

The unsatisfying thing about this result is that perhaps the notion of ordinary space-time can be further abused to countenance a bag that is somehow "ordinary," which surrounds an excluded zone while itself lying within a region of regularity, yet which cannot be extended to a shell.

When the recent interest in dual charge (see, for example, Ezawa and Tze, 1977, and references therein) was communicated to me by L. Halperin, discussion with J. Friedman revealed the following situation, both to myself and J. Arnold, and to Friedman and R. Geroch. [Then we learned of a paper by Clarke (1971), which already covered everything we had proved; that is why this present note omits proofs.] Geroch (1968) had written long ago about the problem of defining a parallel displacement for spinor fields over $(+---)$-Riemannian manifolds. An L-admissible parallel displacement lifts unambiguously to an $SU(2)$-admissible parallel displacement if both displacements are infinitesimal. However, if the displacement is defined jointly in terms of infinitesimal displacements and seaming functions used in switching from one patch to another, the twofold ambiguity in mapping $L \to SU(2)$ can lead to inconsistency. Whether or not a consistent spinor parallel displacement can be defined turns out to depend on the underlying manifold M itself. It is precisely when M possesses a two-sphere that cannot be thickened to an ordinary shell that a difficulty might arise. The two-sphere S in question can be thickened to a four-dimensional neighborhood S', which is homeomorphic to a bundle of disks over the two-sphere S_2, in which either hemisphere of S_2 underlies a tensor product, but where a seaming along the equator involves some integral number n of revolutions of the disk. The shell corresponds to $n = 0$; I will therefore use the term zero-shell, and I will call the other cases n-shells. A spinor displacement is not barred in the more general case of n even; this also corresponds to trivial gravitational dual charge. A spinor displacement is, however, not possible, and nontrivial gravitational dual charge is indeed attained, for n odd. The spin situation has led to the following language: The four-manifolds M where all two-spheres in M thicken to n-shells with n even are called "spin manifolds," while other M are "nonspin manifolds."

If the existence of spin-$\frac{1}{2}$ particles is taken as proof that physical space "is a spin manifold", then gravitational dual charge does not exist in physical space. This seems to me too much of a play on words, however, for the following reason. As Dirac noted long ago, a spinor ψ need not have a well-defined phase where it is zero; indeed $\psi = 0$ "nodal sheets" of unremovable phase singularity play an important role in Dirac's conception of the magnetic monopole. A Dirac equation requires a spinor connection only over that subset M_1 of M where its spinor does not vanish. A spinor ψ can therefore be well defined over a nonspin manifold M, provided that it is zero on such a large subset $Z(\psi)$ that $M_1 = M - Z(\psi)$ is a spin manifold. This consideration reopens the possibility of physical gravitational dual charge, but relates it to Dirac-like nodal regions of spinor fields.

Another consideration, however, shows that it is impossible for a *localized physical entity* to possess nonzero gravitational dual charge. Indeed, this may already seem obvious, inasmuch as this entity cannot be surrounded by a zero-shell. Yet it is a priori conceivable that a one-shell is somehow not different enough from a zero-shell to "attract attention." After all, the only way to tell a Mobius band from a cylindrical band is by a global test.

The situation here is, however, simpler: there is a *local* test to distinguish an "ordinary" two-sphere fence in space-time from two-spheres that thicken to n-shells, $n \neq 0$, if by "ordinary" we imply, "everywhere spacelike,"—namely, the other two-spheres may not be everywhere spacelike. Indeed, if a two-sphere in a Riemannian $+---$ manifold is everywhere spacelike, it is easy to construct a timelike vector field that is everywhere nonzero over the two-sphere: For example, thin an open covering of coordinate patches to a finite subcovering, choose a $(1, 0, 0, 0)$ field in each patch, fade it smoothly to 0 at the patch edges, and add these fields. This is an everywhere transverse smooth vector field, not possible in the case of the n-shells, $n \neq 0$ (Geroch, 1968).

In a sense, gravitational dual charge exists: Choose, e.g., a one-shell, and invest it with a (+− − −)-metric (Markus, 1965). (The Einstein tensor $R_{ij} - \frac{1}{2}g_{ij}R$ will not in general be zero, but it may be taken to compute a source stress tensor. Whether this can be relaxed to 0 is not discussed.)

In another, Cauchy-problem sense, gravitational dual charge does not exist, because given data on an everywhere-spacelike three-space interrupted by spacelike two-spheres that sequester possible monstrosities, none of the monstrosities totals up any gravitational dual charge on its fencing two-sphere: even in a situation with gravitational dual charge present over a (necessarily not-everywhere-spacelike) two-sphere by construction, no *spacelike* two-sphere can *isolate* a gravitational dual charge.

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